

ON THE RATE OF GRADED MODULES

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ABSTRACT. Let K be a field, R a standard graded K -algebra and M be a finitely generated graded R -module. The rate of M , $\text{rate}_R(M)$, is a measure of the growth of the shifts in the minimal graded free resolution of M . In this paper, we find upper bounds for this invariant.

More precisely, let (A, \mathfrak{n}) be a regular local ring and $I \subseteq \mathfrak{n}^t$ be an ideal of A , where $t \geq 2$. We prove that if $(B = A/I, \mathfrak{m} = \mathfrak{n}/I)$ is a Cohen-Macaulay local ring with multiplicity $e(B) = \binom{h+t-1}{h}$, where $h = \text{embdim}(B) - \dim B$, then $\text{Rate}(\text{gr}_{\mathfrak{m}}(B)) = t - 1$ and for every B -module N , which annihilated by a minimal reduction of \mathfrak{m} , $\text{rate}_{\text{gr}_{\mathfrak{m}}(B)}(\text{gr}_{\mathfrak{m}}(N)) \leq t - 1$.

Among other things, we find an upper bound for the rate and regularity of tensor product of modules in term of the rate and regularity of each of them.

INTRODUCTION

Let R be a standard graded K -algebra with the homogeneous maximal ideal \mathfrak{m} and residue field K . There are several invariants attached to a finitely generated graded R -module M . One is the Castelnuovo-Mumford regularity of M , defined by

$$\text{reg}_R(M) := \sup\{t_i^R(M) - i : i \geq 0\},$$

where for all $i \geq 0$, $t_i^R(M)$ denotes the maximum degree of minimal generators of the i th syzygy module of M . This invariant plays an important role in the study of homological properties of M . Regularity of a module can be infinite. Avramov and Peeva in [5] showed that $\text{reg}_R(K)$ is zero or infinite. The K -algebra R is called Koszul if $\text{reg}_R(K) = 0$. From certain point of views, Koszul algebras behave homologically as polynomial rings. Avramov and Eisenbud in [4] showed that if R is Koszul, then the regularity of every finitely generated graded R -module is finite.

Another important invariant is the rate of graded modules. The notion of rate for algebras introduced by Backelin ([6]) and it is generalized in [3] for graded modules. The rate of a finitely generated graded module M over R is defined by

$$\text{rate}_R(M) := \sup\{t_i^R(M)/i : i \geq 1\},$$

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This invariant is always finite (see [3]). The Backelin rate of the algebra R is denoted by $\text{Rate}(R)$ and is equal to $\text{rate}_R(\mathfrak{m}(1))$, the rate of the unique homogenous maximal ideal of R which is shifted by 1. By the definition, one can see that $\text{Rate}(R) \geq 1$ and the equality holds if and only if R is Koszul, so that the $\text{Rate}(R)$ can be taken as a measure how much R deviates from being Koszul. The goal of this paper is to study the rate of modules and find upper bounds for this invariant over some special rings.

This paper is organized as follows:

In section 1 we present some notations and generalities.

Section 2 is devoted to study the behavior of the rate of modules via change of rings. More precisely, let $\varphi : R \rightarrow S$ be a surjective homomorphism of standard graded K -algebras. Then we show that (Theorem 2.4) for every finitely generated graded S -module one has

$$\text{rate}_R(M) \leq \max\{\text{rate}_S(M), \text{rate}_R(S)\} + \max\{0, t_0^S(M)\}.$$

The aim of Section 3 is to study the rate of modules over some special rings. Let (S, \mathfrak{m}) be a complete Cohen-Macaulay local ring. So, it can be written as a quotient of a regular local ring (A, \mathfrak{n}) over a perfect ideal I of A such that $I \subseteq \mathfrak{n}^t$ for some $t \geq 2$. Also, $e(S) \geq \binom{h+t-1}{h}$ (see [15]), where $h = \text{embdim}(S) - \dim S$, and S is called t -extremal if the equality holds. In Theorem 3.6, we prove that if S is t -extremal, then for every finitely generated S -module M annihilated by a minimal reduction of \mathfrak{m} , $\text{rate}_{gr_{\mathfrak{m}}(S)}(gr_{\mathfrak{m}}(M)) \leq t - 1$ and this bound is sharp. In the sense that there is an S -modules M annihilated by a minimal reduction of \mathfrak{m} such that $\text{rate}_{gr_{\mathfrak{m}}(S)}(gr_{\mathfrak{m}}(M)) = t - 1$. This is a generalization of a result proved by the first author in the case where R is 2-external ([2, 2.14]).

An Artinian local ring (R', \mathfrak{n}) is called *stretched* if \mathfrak{n}^2 can be generated by one element. This notion introduced by Sally in [16]. She studied (non-graded) zero dimensional stretched Gorenstein local rings S and determined a defining ideal of R' , when R' is considered as a quotient ring of a regular local ring, with the assumption that the characteristic of R'/\mathfrak{n} is not 2.

In Section 3, we also study the rate of Artinian standard graded stretched K -algebra R with the assumption that K is algebraically closed. In this case the Hilbert series of R is of the form

$$H_R(z) = 1 + hz + z^2 + \cdots + z^s.$$

In Theorem 3.8, we show that if $h, s \geq 2$ then $\text{Rate}(R) = s$. Moreover, if $S = K[X_1, \dots, X_h]$, then $R \simeq S/\text{Lex}(H(R))$, where $\text{Lex}(H(R))$ is the lexsegment ideal of the Hilbert function $H(R)$ of R (see 3.7 for definition). Therefore, the class of all standard graded K -algebras with Hilbert series $H(z) = 1 + hz + z^2 + \cdots + z^s$ has only one element.

Backelin and Fröberg in [7] proved that the tensor product of two Koszul algebras are Koszul. In the last section we are going to extend their result for the rate of modules.

More precisely, in Proposition 4.2 we prove that if R and S are two standard graded K -algebras and M and N are finitely generated graded modules over R and S respectively, then $\text{rate}_T(M \otimes_K N) \leq \max\{\text{rate}_R(M), \text{rate}_S(N)\}$, where $T = R \otimes_K S$ and a consequence of this is the result of Backelin and Fröberg.

1. Notations and Generalities

In this section we prepare some notations and preliminaries which will be used in the paper.

Throughout this paper, unless otherwise stated, K is a field and $R = \bigoplus_{i \in \mathbb{N}_0} R_i$ is a standard graded K -algebra, i.e. $R_0 = K$ and R is generated (as a K -algebra) by finitely many elements of degree one. Also, $M = \bigoplus_{i \in \mathbb{Z}} M_i$ denotes a finitely generated graded R -module.

Remark 1.1.

- (1) For each $d \in \mathbb{Z}$ we denote by $M(d)$ the graded R -module with $M(d)_p = M_{d+p}$, for all $p \in \mathbb{Z}$.

Denote by \mathfrak{m} the maximal homogeneous ideal of R , that is $\mathfrak{m} = \bigoplus_{i \in \mathbb{N}} R_i$. Then we may consider K as a graded R -module via the identification $K = R/\mathfrak{m}$.

- (2) The Hilbert series of M is defined by

$$H_M(z) := \sum_{i \in \mathbb{Z}} \dim_k(M_i) z^i \in \mathbf{Q}[[z]][z^{-1}].$$

- (3) A minimal graded free resolution of M as an R -module is a complex of free R -modules

$$\mathbf{F} = \cdots \rightarrow F_i \xrightarrow{\partial_i} F_{i-1} \rightarrow \cdots \rightarrow F_1 \xrightarrow{\partial_1} F_0 \rightarrow 0$$

such that $H_i(\mathbf{F})$, the i -th homology module of \mathbf{F} , is zero for $i > 0$, $H_0(\mathbf{F}) = M$ and $\partial_i(F_i) \subseteq \mathfrak{m}F_{i-1}$, for all $i \in \mathbb{N}_0$. Each F_i is isomorphic to a direct sum of copies of $R(-j)$, for $j \in \mathbb{Z}$. Such a resolution exists and any two minimal graded free resolutions are isomorphic as complexes of graded R -modules. So, for all $j \in \mathbb{Z}$ and $i \in \mathbb{N}_0$ the number of direct summands of F_i isomorphic to $R(-j)$ is an invariant of M , called the ij -th graded Betti number of M and denoted by $\beta_{ij}^R(M)$.

Also, by definition, the i -th Betti number of M as an R -module, denoted by $\beta_i^R(M)$, is the rank of F_i . By construction, one has $\beta_i^R(M) = \dim_k(\text{Tor}_i^R(M, K))$ and $\beta_{ij}^R(M) = \dim_k(\text{Tor}_i^R(M, K)_j)$.

- (4) For every integer i we set

$$t_i^R(M) := \max\{j : \beta_{ij}^R(M) \neq 0\}.$$

If $\beta_i^R(M) = 0$ we set $t_i^R(M) = 0$.

(5) The Castelnuovo-Mumford regularity of M is defined by

$$\operatorname{reg}_R(M) := \sup\{t_i^R(M) - i : i \in \mathbb{N}_0\}.$$

This invariant is, after Krull dimension and multiplicity, perhaps the most important invariant of a finitely generated graded R -module.

(6) The initial degree $\operatorname{indeg}(M)$ of M is the minimum of i 's such that $M_i \neq 0$. We say that M has d -linear resolution if $\operatorname{indeg}(M) = d$ and $\operatorname{reg}_R(M) = d$.

Definition and Remark 1.2. Following [12], M is called Koszul if the associated graded module $M^g := \bigoplus_{i \geq 0} \mathfrak{m}^i M / \mathfrak{m}^{i+1} M$ has 0-linear resolution. The ring R is Koszul if the residue field K , as an R -module, is Koszul.

The Castelnuovo-Mumford regularity plays an important role in the study of homological properties of M and it is clear that $\operatorname{reg}_R(M)$ can be infinite. Avramov and Peeva in [5] proved that $\operatorname{reg}_R(K)$ is zero or infinite. Also, Avramov and Eisenbud in [4] showed that if R is Koszul, then the regularity of every finitely generated graded R -module is finite.

2. THE RATE OF MODULES

In this section we study the rate of some graded modules. The notion of rate for algebras introduced by Backelin in [6] and generalized in [3] for graded modules. The rate of the graded algebra R is an invariant that measures how far R is from being Koszul.

Definition and Remark 2.1.

(1) The Backelin rate of R is defined as

$$\operatorname{Rate}(R) := \sup\{(t_i^R(K) - 1)/i - 1 : i \geq 2\},$$

and generalization of this for modules is defined by

$$\operatorname{rate}_R(M) := \sup\{t_i^R(M)/i : i \geq 1\}.$$

(2) A comparison with Backelin's rate shows that $\operatorname{Rate}(R) = \operatorname{rate}_R(\mathfrak{m}(1))$. Also, it turns out that the rate of any module is finite (see [3, 1.3]).

(3) One can see that $\operatorname{rate}_R(M) \leq \operatorname{reg}_R(M) + 1$.

Remark 2.2. Consider a minimal presentation of R as a quotient of a polynomial ring, i.e.

$$R \cong S/I,$$

where $S = k[X_1, \dots, X_n]$ is a polynomial ring and I is an ideal generated by homogeneous elements of degree ≥ 2 . I is called a defining ideal of R . Let $m(I)$ denotes the maximum of

the degree of a minimal homogeneous generator of I . It follows from (the graded version of) [8, 2.3.2] that $t_2^R(K) = m(I)$. Thus one has

$$\text{Rate}(S/I) \geq m(I) - 1.$$

From the above inequality one can see that $\text{Rate}(R) \geq 1$ and the equality holds if and only if R is Koszul. So that $\text{Rate}(R)$ can be taken as a measure of how much R deviates from being Koszul. Also, for a module M with $\text{indeg}(M) = t_0^R(M) = 0$ we have $\text{rate}_R(M) \geq 1$ and the equality holds if and only if M is Koszul.

The following lemma will be used in the rest of this paper.

Lemma 2.3. *Let*

$$\cdots \rightarrow L_n \rightarrow L_{n-1} \rightarrow \cdots \rightarrow L_1 \rightarrow L_0 \rightarrow L \rightarrow 0$$

be an exact sequence of graded R -modules and homogeneous homomorphisms. Then for all $j \in \mathbb{Z}$

$$t_n(L) \leq \max\{t_{n-i}(L_i); 0 \leq i \leq n\}.$$

Proof. We prove the claim by induction on n .

In the case $n = 0$, the result follows using the surjection

$$\text{Tor}_0(L_0, K)_j \rightarrow \text{Tor}_0(L, K)_j$$

and Remark 1.1(3). Now, let $n > 0$ and suppose that the result has been proved for smaller values of n . let K_1 be the kernel of the homomorphism $L_0 \rightarrow L$. Then, using the exact sequence

$$\cdots \rightarrow L_i \rightarrow \cdots \rightarrow L_1 \rightarrow K_1 \rightarrow 0$$

and the inductive hypothesis, we have

$$(2.1) \quad t_{n-1}(K_1) \leq \max\{t_{n-1-i}(L_{i+1}) | 0 \leq i \leq n-1\}.$$

Now, the result follows using 2.1 and the exact sequence

$$0 \rightarrow K_1 \rightarrow L_0 \rightarrow L \rightarrow 0.$$

□

In the following theorem we study the rate of modules via change of rings.

Theorem 2.4. *Let $\varphi : R \rightarrow S$ be a surjective homomorphism of standard graded K -algebras. Assume that M is a finitely generated graded S -module. Then*

$$\text{rate}_R(M) \leq \max\{\text{rate}_S(M), \text{rate}_R(S)\} + \max\{0, t_0^S(M)\}.$$

Proof. Let

$$\mathbf{F} : \cdots \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow 0$$

be the minimal graded free resolution of M as an S -module. Applying Lemma 2.3 to \mathbf{F} , one has

$$t_n^R(M) \leq \max\{t_0^R(F_n), t_1^R(F_{n-1}), \dots, t_n^R(F_0)\}.$$

Note that

$$t_i^R(F_j) = t_i^R(\oplus_{r \in \mathbb{Z}} S(-r)^{\beta_{jr}^S(M)}) = t_i^R(S) + t_j^S(M).$$

In particular, since $t_0^R(S) = 0$, we have $t_0^R(F_j) = t_j^S(M)$. Therefore, for any integer $n \geq 1$, we get

$$(2.2) \quad \frac{t_n^R(M)}{n} \leq \max\left\{\frac{t_n^S(M)}{n}, \frac{t_1^R(S) + t_{n-1}^S(M)}{n}, \dots, \frac{t_n^R(S) + t_0^S(M)}{n}\right\}$$

Let $a = \max\{\text{rate}_R(S), \text{rate}_S(M)\}$ then for all $j > 0$, we have $t_j^S(M) \leq ja$ and $t_j^R(S) \leq ja$. Now by (2.2), one has

$$\frac{t_n^R(M)}{n} \leq \max\{a, a + t_0^S(M)\}$$

and this implies our desired inequality. \square

Remark 2.5. *Let the situation be as in the above theorem.*

- (1) *Aramova et al. in [3, 1.2] studied the behavior of rate via change of rings. They showed that*

$$\text{rate}_S(M) \leq \max\{\text{rate}_R(M), \text{rate}_R(S)\}.$$

But, one can see that their proof works for the case where M is non-negatively graded over S .

- (2) *We remark that in particular case in the above theorem, if $t_0^S(M) \leq 0$ and $\text{rate}_R(S) = 1$, then*

$$\text{rate}_R(M) \leq \text{rate}_S(M).$$

- (3) *In view of (1) and (2), if M is generated in degree zero and $\text{rate}_R(S) = 1$, then*

$$\text{rate}_R(M) = \text{rate}_S(M).$$

3. RATE OF MODULES OVER SOME SPECIAL RINGS

In this section we are going to study the rate of modules over some special class of rings.

There are various methods for proving that a K -algebra is Koszul. One of them is the concept of Koszul filtration. This notion, introduced in [11] by Conca, Valla and Trung and extended by Conca, Negri and Rossi in [9] in order to study the rate of graded algebras.

Definition 3.1. A family \mathbf{F} of ideals of R is said to be a generalized Koszul filtration of R if

- (1) the ideal 0 and the maximal homogeneous ideal \mathfrak{m} of R belong to \mathbf{F} ,
- (2) for every $I \in \mathbf{F}$ different from 0 there exists $J \in \mathbf{F}$ such that $J \subseteq I$, I/J is cyclic, $J : I \in \mathbf{F}$ and $m(J) \leq m(I)$.

In general the existence of a generalized Koszul filtration is a strong condition. We take advantage of the existence of generalized Koszul filtration for algebras to determine the rate of modules over them.

The following proposition, which is shown in [9, proposition 1.2], finds an upper bound for $\text{Rate}(R)$.

Proposition 3.2. Let \mathbf{F} be a generalized Koszul filtration of R and let d be an integer such that $m(I) \leq d$ for every ideal $I \in \mathbf{F}$. Then

$$\text{Rate}(R) \leq d.$$

The following two lemmas will be used in the next theorem.

Lemma 3.3. Let d be a positive integer and l be a homogeneous element of R of degree $\leq d$. Assume that one of the following conditions holds

- (1) $(0 : l) = 0$,
- (2) $(0 : l) = \mathfrak{m}$ where \mathfrak{m} is the graded maximal ideal of R ,
- (3) $(0 : l) = (l)$.

Then any generalized Koszul filtration \mathcal{F} of R/lR with $m(I) \leq d$ for each $I \in \mathcal{F}$, can be lifted to a filtration \mathcal{F}' of R such that $m(J) \leq d$ for each $J \in \mathcal{F}'$.

Proof. Let \mathcal{F} be a generalized Koszul filtration of R/lR . Set

$$\mathcal{F}' = \{I \subset R : l \in I \text{ and } I/(l) \in \mathcal{F}\}.$$

Then $(l) \in \mathcal{F}'$ and by the assumption it is clear that $(0 : l) \in \mathcal{F}'$. Now, one can see that \mathcal{F}' satisfies the conditions of Definition 3.1. \square

Using the definition of the concept of rate, it is straightforward to see that:

Lemma 3.4. Let R be an Artinian standard graded K -algebra such that $R_i = 0$ for all $i \geq t$. Then for a finitely generated graded R -module M ,

$$\text{rate}_R(M) \leq t_0(M) + t - 1.$$

Definition and Remark 3.5. Let (R, \mathfrak{m}) be a (non-graded) Cohen-Macaulay complete local ring. Let $R \simeq A/I$ be a minimal Cohen presentation of R with (A, \mathfrak{n}) a regular local ring and $I \subseteq \mathfrak{n}^t$ a perfect ideal of A with $t \geq 2$. Then it is well-known that $e(R) \geq h + 1$ where $e(R)$ is the multiplicity of R and $h = \text{embdim}(R) - \dim R$ ([1]). The ring R is called of minimal multiplicity if the equality holds.

The first author in [2] showed that if R is of minimal multiplicity, then $R^g := \text{gr}_{\mathfrak{m}}(R)$ has a Koszul filtration and every R module annihilated by a minimal reduction of \mathfrak{m} is a Koszul module.

As remarked in [15], if $I \subseteq \mathfrak{n}^t$ with $t \geq 3$ the inequality $e(R) \geq h + 1$ is not sharp. Let $x_1, \dots, x_d \in \mathfrak{n}$ such that their images in R generate a minimal reduction of \mathfrak{m} . Then

$$e(R) = \ell(\bar{A}/\bar{\mathfrak{n}}^t) + \ell(\bar{\mathfrak{n}}^t/\bar{I}) = \binom{h+t-1}{h} + \ell(\bar{\mathfrak{n}}^t/\bar{I}),$$

where “ $-$ ” denotes the image of the natural homomorphism $A \rightarrow A/(x_1, \dots, x_d)$. Thus $e(R) \geq \binom{h+t-1}{h}$ and the equality hold if and only if $\bar{I} = \bar{\mathfrak{n}}^t$. In this case I is called t -extremal.

Rossi and Valla in [15] proved that if I is t -external, then R^g is Cohen-Macaulay and $v(I) = \binom{h+t-1}{t} = v(I^*)$, where I^* is the ideal generated by the initial forms of elements of I and $v(J)$ denotes the minimal number of generators of an ideal J .

In the next theorem we study the rate of modules over these class of rings.

Theorem 3.6. Let (R, \mathfrak{m}, k) be a (non-graded) Cohen-Macaulay local ring and $e(R) = \binom{h+t-1}{h}$ where t is the initial degree of a defining ideal of R^g and $h = \text{embdim}(R) - \dim R$. Let M be an R -module and J be a minimal reduction of \mathfrak{m} . Then

- (1) $\text{Rate}(R^g) = t - 1$,
- (2) if $JM = 0$, then $\text{rate}_{R^g}(M^g) \leq t - 1$.
In particular, if $t_1^{R^g}(M^g) = t - 1$ then $\text{rate}_{R^g}(M^g) = t - 1$.

Proof. We may assume that the residue field K is infinite. Indeed, let (S, \mathfrak{n}) be a Noetherian local ring and N be a finitely generated S -module. Assume that X is a variable over S . Set $T = S[X]_{\mathfrak{n}[X]}$. Then the natural map $S \rightarrow T$ is a faithfully flat extension of Noetherian local rings of the same Krull dimension and embedding dimension. The ideal J of S is a minimal reduction of \mathfrak{n} if and only if JT is a minimal reduction of $\mathfrak{n}S$ and S is Cohen-Macaulay if and only if T is Cohen-Macaulay (see [13]).

Also, note that if \mathfrak{u} denotes the maximal ideal of T then, $T/\mathfrak{u}T$ is an infinite field and one can see that $S^g \otimes_{S/\mathfrak{n}} T/\mathfrak{u}T \simeq T^g$, as graded algebras over the field $T/\mathfrak{u}T$. Therefore, $e(S) = e(T)$ and the initial degree of the defining ideal of S^g and T^g are the same. Also, by setting $L = N[X]_{\mathfrak{n}[X]}$ one has $N^g \otimes_{S/\mathfrak{n}} T/\mathfrak{u}T \simeq L^g$, as graded T^g -modules. Since $S/\mathfrak{n} \rightarrow T/\mathfrak{u}T$ is faithfully flat extension of the residue fields we have $t_i^{S^g}(N^g) = t_i^{T^g}(L^g)$, for all $i \geq 0$.

After reducing to infinite residue field, one can see that we may also reduce to the case where R is complete with respect to the \mathfrak{m} -adic filtration.

Therefore, there exists a regular local ring (A, \mathfrak{n}) and a perfect ideal I of codimension h such that $I \subseteq \mathfrak{n}^t$, for some $t \geq 2$, and $R = A/I$. By [15, theorem 3.3], $R^g = k[X_1, \dots, X_{d+h}]/I^*$ is Cohen-Macaulay with $v(I) = v(I^*) = \binom{h+t-1}{t}$. Now, since k is infinite, we may assume that X_{h+1}, \dots, X_{d+h} is a regular sequence modulo I^* . Since $v(I^*) = \binom{h+t-1}{t}$ and $\text{indeg}(I^*) = t$, we get

$$R^g/(\overline{X}_{h+1}, \dots, \overline{X}_{d+h}) \cong k[X_1, \dots, X_h]/(X_1, \dots, X_h)^t.$$

Then the set

$$\mathcal{F} = \{U \subset R^g/(\overline{X}_{h+1}, \dots, \overline{X}_{d+h}) : U \text{ is a monomial ideal and } m(U) \leq t-1\}$$

is a generalized Koszul filtration for $R^g/(\overline{X}_{h+1}, \dots, \overline{X}_{d+h})$, (see [9, 1.4]). By Lemma 3.3, we conclude that R^g has a generalized Koszul filtration \mathcal{F}' such that for each $J \in \mathcal{F}'$, $m(U) \leq t-1$. Therefore, using Proposition 3.2, we get

$$\text{Rate}(R^g) \leq t-1.$$

By assumption $\text{indeg}(I^*) = t$ and then $t \leq m(I^*)$. Now, by Remark 2.2 we get $\text{Rate}(R^g) = t-1$.

(2) We use the above notations and set $S := k[X_1, \dots, X_h]/(X_1, \dots, X_h)^t$. Then $S_i = 0$ for all $i \geq t$. Now, by Lemma 3.4, we have

$$\text{rates}(M^g) \leq t-1.$$

Since $\overline{X}_{h+1}, \dots, \overline{X}_{d+h}$ is an R -regular sequence, the Koszul complex with respect to this sequence is a minimal graded free resolution of $R/(\overline{X}_{h+1}, \dots, \overline{X}_{d+h})$ and obviously is a 0-linear resolution. Therefore, $\text{rate}_R(S) = 1$. This, in conjunction with the fact that M^g is generated in degree zero and Remark 2.5(3), implies that

$$\text{rate}_{R^g}(M^g) \leq t-1,$$

as we desired. □

An Artinian local ring (R', \mathfrak{n}) is called *stretched* if \mathfrak{n}^2 can be generated by one element. This notion introduced by Sally in [16]. She studied zero dimensional stretched Gorenstein local rings S and determined a defining ideal of R' when R' is considered as a quotient ring of a regular local ring with the assumption that the characteristic of R'/\mathfrak{n} is not 2.

In the next theorem we determine the rate of Artinian standard graded stretched K -algebras with the assumption that K is algebraically closed. Such an algebra has Hilbert series of the form

$$H_R(z) = 1 + hz + z^2 + \cdots + z^s.$$

Definition and Remark 3.7. Let $S = K[X_1, \dots, X_n] = \bigoplus_{d \geq 0} S_d$ be a polynomial ring over a field K which is graded in the usual way.

- (1) We denote by $\text{Mond}(S)$ the set of all monomials of S of degree d and $>_{\text{lex}}$ the degree lexicographic order on $\text{Mon}(S)$. A set $L \subseteq \text{Mond}(S)$ of monomials is called a *lexsegment* if there exists $u \in L$ such that $v \in L$ for all $v \in \text{Mond}(S)$ with $v >_{\text{lex}} u$. A monomial ideal I is said to be a *lex segment ideal* if for each d the monomials of degree d in I form a *lexsegment* set of monomials.
- (2) Given a standard graded K -algebra $R = S/I = \bigoplus_{d \geq 0} R_d$, let us consider the Hilbert function

$$H(R)(d) := \dim_K(R_d).$$

It is proved in [14] that if, for every $d \geq 0$ we leave out the smallest $H(R)(d)$ monomials in S_d , the remaining monomials generate a monomial ideal J which is a *lex segment ideal* with the property that

$$H(S/J) = H(R).$$

Since the ideal J is uniquely determined by the Hilbert function $H(R)$, it is called the *lex segment ideal* associated to the Hilbert function $H(R)$ and is denoted by $\text{Lex}(H(R))$.

Theorem 3.8. Let R be a standard graded algebra over an algebraically closed field K . Assume that I is a defining ideal of R and the Hilbert series of R is of the form $H_R(z) = 1 + hz + z^2 + \cdots + z^{s+1}$ with $h, s \geq 2$. Then

- (1) $\tau(R) := \dim_K(0 : R_1) = h$,
- (2) $\text{Rate}(R) = s + 1$,
- (3) $m(I) = s + 2$,
- (4) $I = \text{Lex}(H(R))$.

Proof. First, we prove 1) – 3) by induction on h . Since R is standard, $R_1 R_i = R_{i+1} \neq 0$ for $2 \leq i \leq s$. This implies that $\tau(R) \leq h$. Let $h = 2$. Since $\dim R_1 > \dim R_2$, by [10, Lemma 2.8], there exists a non-zero linear form l such that $l^2 = 0$. If $lR_1 \neq 0$, then $lR_1 = R_2$. So

$$R_3 = R_1 R_2 = R_1 (lR_1) = lR_2 = l(lR_1) = 0,$$

which is a contradiction. Therefore, $lR_1 = 0$ and this, in conjunction with $R_1 R_{s+1} = R_{s+2} = 0$, implies that $\tau(R) = 2$. By the assumption, $R_i = 0$ for $i \geq s + 2$. So, using Lemma 3.4,

$$m(I) - 1 \leq \text{Rate}(R) \leq s + 1.$$

Therefore, $m(I) \leq s + 2$. On the other hand, since $l \in (0 : R_1)$,

$$H_{R/lR}(z) = 1 + z + z^2 + \cdots + z^{s+1}.$$

It follows that

$$R/lR \simeq K[x]/(x^{s+2}).$$

Thus $m(I) = s + 2$ and we get $\text{Rate}(R) = s + 1$.

Now, let $h > 2$ and suppose that the result has been proved for smaller values of h . Then there exists a non-zero linear form l such that $l^2 = 0$. Therefore, one can see that

$$l \in (0 : R_1) \quad \text{and} \quad ((l) : R_1) = (0 : R_1).$$

Hence, we get

$$H_{R/lR}(z) = 1 + (h - 1)z + z^2 + \cdots + z^{s+1}$$

and

$$\tau(R/lR) = \tau(R) - 1.$$

By inductive hypothesis, $\tau(R/lR) = h - 1$ and then $\tau(R) = h$. Now, let J be the defining ideal of R/lR . Then, by induction, $m(J) = s + 2$. So, $m(I) \geq s + 2$ and, by Lemma 3.4,

$$s + 1 \leq m(I) - 1 \leq \text{Rate}(R) \leq s + 1.$$

Therefore,

$$\text{Rate}(R) = s + 1 \quad \text{and} \quad m(I) = s + 2.$$

4) Using induction on h , one can see that $\dim((0 : R_1)_1) = h - 1$. Therefore, in view of the fact that $\tau(R) = h$, there is a minimal presentation $R = K[X_1, \dots, X_h]/I$ such that $X_i + I \in (0 : R_1)$ for $1 \leq i \leq h - 1$. From this we get

$$I = (\{X_i X_j, X_h^{s+2} : 1 \leq i \leq h - 1, 1 \leq j \leq h\}).$$

Thus $I = \text{Lex}(H(R))$.

□

Remark 3.9. It follows from part 4) of the above theorem that in the case where K is algebraically closed for fixed integers $s, h \geq 2$, the class of all standard graded K -algebras with Hilbert series $H(z) = 1 + hz + z^2 + \cdots + z^s$, has only one element.

4. RATE AND REGULARITY OF TENSOR PRODUCT OF MODULES

In this section we find upper bounds for the rate and regularity of tensor product of modules over standard graded K -algebras in terms of rate and regularity of those modules.

The following lemma will be used in the next theorem.

Lemma 4.1. *Let R and S be two standard graded K -algebra and M and N be two finitely generated graded R and S -modules, respectively. Set $T := R \otimes_K S$. Then*

$$t_n^T(M \otimes_K N) \leq \max\{t_i^R(M) + t_j^S(N); i, j \geq 0 \text{ and } i + j = n\}.$$

Proof. Let

$$\mathbf{F} : \cdots \rightarrow F_i \rightarrow F_{i-1} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0$$

and

$$\mathbf{G} : \cdots \rightarrow G_j \rightarrow G_{j-1} \rightarrow \cdots \rightarrow G_0 \rightarrow N \rightarrow 0$$

be the minimal graded free resolution of M and N as an R and S -module, respectively. Then applying $- \otimes_K N$ to the complex \mathbf{F} and $M \otimes_K -$ to the complex \mathbf{G} we obtain two exact complex of graded T -modules

$$(4.1) \quad \mathbf{F} \otimes_K N : \cdots \rightarrow F_i \otimes_K N \rightarrow \cdots \rightarrow F_0 \otimes_K N \rightarrow M \otimes_K N \rightarrow 0$$

and

$$M \otimes_K \mathbf{G} : \cdots \rightarrow M \otimes_K G_j \rightarrow \cdots \rightarrow M \otimes_K G_0 \rightarrow M \otimes_K N \rightarrow 0.$$

Note that $R \otimes_K \mathbf{G}$ is an exact complex of graded free T -modules and one can see that, actually, $R \otimes_K \mathbf{G}$ is the minimal graded free resolution of $R \otimes_K N$ as a T -module.

For all $j \in \mathbb{N}_0$ let $G_j = \bigoplus_{r \in \mathbb{Z}} S(-r)^{\beta_{jr}^S(N)}$. Then $R \otimes_K G_j \cong \bigoplus_{r \in \mathbb{Z}} T(-r)^{\beta_{jr}^S(N)}$ and hence

$$t_j^T(R \otimes_K N) = t_j^S(N).$$

Therefore, for all $i, j \in \mathbb{N}_0$,

$$\begin{aligned} t_j^T(F_i \otimes_K N) &= t_j^T(\bigoplus_{r \in \mathbb{Z}} R(-r)^{\beta_{ir}^R(M)} \otimes_K N) \\ &= \max\{t_j^T(R(-r) \otimes_K N) : \beta_{ir}^R(M) \neq 0\} \\ &= \max\{t_j^T(R \otimes_K N) + r : \beta_{ir}^R(M) \neq 0\} \\ &= t_j^T(R \otimes_K N) + t_j^R(M) \\ &= t_j^S(N) + t_i^R(M). \end{aligned}$$

Now, using the exact sequence 4.1 and Lemma 2.3, we have

$$\begin{aligned} t_n^T(M \otimes_K N) &\leq \max\{t_{n-i}^T(F_i \otimes_K N); 0 \leq i \leq n\} \\ &= \max\{t_j^S(N) + t_i^R(M); i, j \geq 0 \text{ and } i + j = n\}. \end{aligned}$$

□

The following theorem finds an upper bound for the rate and regularity of tensor product of two modules in terms of the rate and regularity of each of them. Also, it gives a sufficient condition for the Koszulness of tensor product of two algebras.

Theorem 4.2. *Let the situations be as in the above theorem. Then*

- (1) $\text{rate}_T(M \otimes_K N) \leq \max\{\text{rate}_R(M), \text{rate}_S(N)\}$. In particular, if R and S are Koszul algebras, then so is $R \otimes_K S$.
- (2) $\text{reg}_T(M \otimes_K N) \leq \text{reg}_R(M) + \text{reg}_S(N)$.

Proof. 1) Let $\text{rate}_R(M) = a$ and $\text{rate}_S(N) = b$. Then, by definition, for all $i, j \geq 1$, we have

$$t_i^R(M) \leq ai \quad \text{and} \quad t_j^S(N) \leq bj.$$

So, if $i + j = n$ then

$$t_i^R(M) + t_j^S(N) \leq ai + bj \leq \max\{a, b\}n.$$

Now, by the above lemma,

$$t_n^T(M \otimes_K N) \leq \max\{a, b\}n.$$

Therefore,

$$\text{rate}_T(M \otimes_K N) \leq \max\{\text{rate}_R(M), \text{rate}_S(N)\},$$

as desired.

- 2) The proof is straightforward, using the above lemma. □

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